Chapter 3

Distribution of the Risk Factor

Generally, it is claimed, for the purpose of risk measurement, that the risk factors have the normal distribution. The mathematical reason behind this choice is explained in this section of the paper.

Suppose that the market rate process is denoted by \( (X_t)_{t \geq 0} \). Following the general assumption of finance (Hull, 1993), the market rate process \( (X_t)_{t \geq 0} \) typically conforms to the geometric Brownian motion, that is:

\[
dX_t = \mu_X X_t dt + \sigma_X X_t dB_t
\]  

(1.2)

In the above expression (1.2), \( \mu_X \) denotes the drift rate, \( \sigma_X \) denotes the volatility and \( (B_t)_{t \geq 0} \) denotes the standard Brownian motion. For \( X_0 = x_0 \), this expression will become
\[
X_t = x_0 + \int_0^t \mu X_s \, dS + \int_0^t \sigma X_s \, dB_s
\]

Hence, by applying the Itô’s lemma to \(g(X_t) = \log(X_t)\), the expression becomes

\[
\log(X_t) = \log(X_0) + \int_0^t \frac{1}{X_s} \, dX_s - \frac{1}{2} \int_0^t \frac{\sigma^2 X_s^2}{X_s^2} \, dS
\]

Now, by applying equations (1.2), we get:

\[
X_t = x_0 \exp\left(\left(\mu X - \frac{\sigma^2 X^2}{2}\right)T + \sigma X B_t\right)
\]

(1.3)

It can be easily verified that the equation (1.3) provides the unique solution to the problem in equation (1.2) (Lamberton and Lapeyre, 1993). Thus, the log-returns \(\log(X_t/X_0)\) have the normal distribution with the mean of \((\mu X - \frac{\sigma^2 X^2}{2})T\) and variance \(\sigma^2 X^2\). It is important to note that numerous researches have proved that the hypothesis of the log-returns with normal distribution is merely a rough approximation of the actual scenario (Taylor, 1986). Nevertheless, this hypothesis is extensively employed in practice since the normal distribution has the characteristic of analytical tractability.

By using the typical notation, in which \(\psi_i\) represents the value of the stochastic process at the end of the holding period \(T\) and \(\psi_0^i\) represents the value of the stochastic process at time 0, the equation (1.3) becomes

\[
\psi_i = \psi_0^i \exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)T + \sigma_i B_t^i\right).
\]

In the above equations, \(\mu_i\) denotes the drift, \(\sigma_i\) denotes the volatility and \((B_t^i)_{t \geq 0}\) represents the conventional Brownian motion for the market rate \(i\) (where \(i = 1, \ldots, M\)). This leads to the equation below:
\[
\log \left( \frac{\psi_i}{\psi^0_i} \right) = \left( \mu_i - \frac{\sigma^2_i}{2} \right) T + \sigma_i B^i_T
\] (1.4)

The following expression is obtained by means of first order Taylor expansion around \( \psi^0 \).

\[
\log \left( \frac{\psi_i}{\psi^0_i} \right) = \log \left( \frac{\psi^0_i + (\psi_i - \psi^0_i)}{\psi^0_i} \right) - \log \left( \frac{\psi^0_i}{\psi^0_i} \right)
\approx \left[ \log (\psi^0_i) + \frac{\psi_i - \psi^0_i}{\psi^0_i} \right] - \log (\psi^0_i)
\approx \frac{\psi_i - \psi^0_i}{\psi^0_i}
\]

The above expression along with the equation (1.4) implies that:

\[
\frac{\psi_i - \psi^0_i}{\psi^0_i} \approx \left( \mu_i - \frac{\sigma^2_i}{2} \right) T + \sigma_i B^i_T
\]

By taking this approximation as an absolute equality, the normal distribution of the risk factors is attained.

**Theorem 1.9.** We have \( \psi_i \sim N \left( 0, (\psi^0_i \sigma_i)^2 T \right) \) if \( \frac{\psi_i - \psi^0_i}{\psi^0_i} = \left( \mu_i - \frac{\sigma^2_i}{2} \right) T + \sigma_i B^i_T \)

**Proof.** As the standard Brownian motion is given by \( (B^i_t)_{t \geq 0} \), therefore, \( B^i_t \sim N(0, T) \). This leads to:

\[
\frac{\psi_i - \psi^0_i}{\psi^0_i} \sim N \left( \left( \mu_i - \frac{\sigma^2_i}{2} \right) T, \sigma_i^2 T \right)
\] (1.5)

Hence, we have \( \psi \sim N \left( \psi^0_i \left[ 1 + \left( \mu_i - \frac{\sigma^2_i}{2} \right) T \right], (\psi^0_i \sigma_i)^2 T \right) \) since \( \psi^0_i \) is a constant.
Figure 1.1: Interpretation of Value-at-Risk (Duffie and Pan, 1997).

should also have a normal distribution. The outcome conforms immediately since

\[ E(a^T \omega) = a^T E(\omega) \text{ and } \text{Cov}(a^T \omega) = a^T \text{Cov}(\omega)a \]

**Theorem 1.16.** The VaR of a linear portfolio \( V_i(\omega) = a^T \omega \) corresponds to the VaR(\( \alpha \)) = \( z_\alpha \sqrt{a^T \Sigma a} \), at which \( z_\alpha \) represents the \( \alpha \)-quantile of the standard normal distribution.

**Proof.** Suppose that \( Y \sim N(0, \sigma^2) \), so \( \frac{Y}{\sigma} \sim N(0, 1) \) and \( (−z_\alpha \sigma) \) denotes the \((1 − \alpha)\)-quantile of \( Y \).

**Example 1.17.** The risk profiles of the example portfolio are shown in the Figure 1.2, given below, in order to depict various elements of risk measurement techniques. The figure 1.2 uses the example of an equity portfolio that have seven risk factors \( \omega_1, \ldots, \omega_7 \), each of which denotes an equity index. The dots in the figure illustrate the profit and loss when each of the risk factors shifts one up or down, in the respective order two standard deviations separately. The solid line in the figure demonstrate the quadratic functions \( v^{(i)}(\omega_i) = \frac{1}{2} G_{i,j} \omega_j^2 + \)
Bibliography


