\[ \sigma^2_{t+1} = (1 - \alpha - \beta)\sigma^2_t + \alpha R_t^2 + \beta \sigma^2_t = \sigma^2 + \alpha (R_t^2 - \sigma^2) + \beta (\sigma^2_t - \sigma^2), \]

**Expected Shortfall (ES)**

As the most regularly used quantile-based risk valuation, Value-at-Risk has been intensely belittled. First, VaR does not indicate the dimension the prospective reduction given that this reduction surpasses the VaR. Second, as Artzner et al. (1997, 1999) showed that the VaR is not actually sub-additive. That is, the complete VaR of a portfolio may be higher than the sum of personal VaRs. This may cause problems if the risk control system of a financial institution is depending on VaR limits of personal guides. To get over these inadequacies, Artzner et al. (1997) introduced Expected Shortfall (ES) as an alternative risk evaluate which is not only coherent but also indicators the level of the reduction when a VaR is surpassed.

The Expected Shortfall can be calculated by the following formula:

\[ \text{ES}_\alpha(X) = E[-X | -X \geq \text{VaR}_\alpha(X)]. \]

3. **Distribution of the Risk Factor**

Generally, it is claimed, for the purpose of risk measurement, that the risk factors have the normal distribution. The mathematical reason behind this choice is explained in this section of the paper.

Suppose that the market rate process is denoted by \((X_t)_{t\geq0}\). Following the general assumption of finance (Hull, 1993), the market rate process \((X_t)_{t\geq0}\) typically conforms to the geometric Brownian motion, that is:

\[ dX_t = \mu_t X_t dt + \sigma_t X_t dB_t \]  \hspace{1cm} (1.2)
In the above expression (1.2), \( \mu_x \) denotes the drift rate, \( \sigma_x \) denotes the volatility and \((B_t)_{t \geq 0}\) denotes the standard Brownian motion. For \( X_0 = x_0 \), this expression will become

\[
X_t = x_0 + \int_0^t \mu_x X_s dS + \int_0^t \sigma_x X_s dB_s
\]

Hence, by applying the Itô’s lemma to \( g(X_t) = \log(X_t) \), the expression becomes

\[
\log(X_t) = \log(X_0) + \int_0^t \frac{1}{X_s} dX_s - \frac{1}{2} \int_0^t \frac{\sigma_x^2 X_s^2}{X_s^2} ds
\]

Now, by applying equation (1.2), we get:

\[
X_t = x_0 \exp \left( \left( \mu_x - \frac{\sigma_x^2}{2} \right) t + \sigma_x B_t \right). \tag{1.3}
\]

It can be easily verified that the equation (1.3) provides the unique solution to the problem in equation (1.2) (Lamberton and Lapeyre, 1993). Thus, the log-returns \( \log \left( \frac{X_t}{X_0} \right) \) have the normal distribution with the mean of \( (\mu_x - \frac{\sigma_x^2}{2}) t \) and variance \( \sigma_x^2 \). It is important to note that numerous researches have proved that the hypothesis of the log-returns with normal distribution is merely a rough approximation of the actual scenario (Taylor, 1986). Nevertheless, this hypothesis is extensively employed in practice since the normal distribution has the characteristic of analytical tractability.

By using the typical notation, in which \( \psi_i \) represents the value of the stochastic process at the end of the holding period \( T \) and \( \psi^0_i \) represents the value of the stochastic process at time 0, the equation (1.3) becomes

\[
\psi_i = \psi^0_i \exp \left( \left( \mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i B^i_T \right).
\]
In the above equation, $\mu_i$ denotes the drift, $\sigma_i$ denotes the volatility and $(B^i_t)_{t \geq 0}$ represents the conventional Brownian motion for the market rate $i$ (where $i = 1, \ldots, M$). This leads to the equation below:

$$\log\left(\frac{\psi_i}{\psi^0_i}\right) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)T + \sigma_i B^i_T$$  \hspace{1cm} (1.4)

The following expression is obtained by means of first order Taylor expansion around $\psi^0$.

$$\log\left(\frac{\psi_i}{\psi^0_i}\right) = \log (\psi^0_i + (\psi_i - \psi^0_i)) - \log (\psi^0_i) \\ \approx \left[\log(\psi^0_i) + \frac{\psi_i - \psi^0_i}{\psi^0_i}\right] - \log (\psi^0_i) \\ \approx \frac{\psi_i - \psi^0_i}{\psi^0_i}$$

The above expression along with the equation (1.4) implies that:

$$\frac{\psi_i - \psi^0_i}{\psi^0_i} \approx \left(\mu_i - \frac{\sigma_i^2}{2}\right)T + \sigma_i B^i_T$$

By taking this approximation as an absolute equality, the normal distribution of the risk factors is attained.

**Theorem 1.9:**

We have $\psi_i \sim N(0, (\psi^0_i \sigma_i)^2 T)$ if $\frac{\psi_i - \psi^0_i}{\psi^0_i} = \left(\mu_i - \frac{\sigma_i^2}{2}\right)T + \sigma_i B^i_T$

**Proof:**

As the standard Brownian motion is given by $(B^i_t)_{t \geq 0}$, therefore, $B^i_T \sim N(0, T)$. This leads to:

$$\frac{\psi_i - \psi^0_i}{\psi^0_i} \sim N\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)T, \sigma_i^2 T\right)$$  \hspace{1cm} (1.5)
**Definition 1.14:**

“The Value-at-Risk, for a continuous, firmly augmenting profit and loss distribution function, for a confidence level $\alpha$ is defined as that level of loss $\text{VaR}(\alpha)$ at which:

$$P \left( \{ \omega \in O \mid v(\omega) \leq \text{VaR}(\alpha) \} \right) = 1 - \alpha$$” (Alexander, 1996).

Considering that $v(\omega)$ denotes a random variable instead of representing a deterministic function of a random variable, the VaR conforms to the $(1-\alpha)$-quantile of the profit and loss distribution. This explanation also complements the technique with which the VaR is evaluated:

Firstly, the profit and loss distribution is constructed with the help of the distribution of the risk factors and the deterministic profit and loss function $v(\omega)$. After which, the Value-at-Risk is acquired from the $(1-\alpha)$-quantile of this one–dimensional distribution.

![Figure 1.1: Interpretation of Value-at-Risk (Duffie and Pan, 1997).](image)

**Value-at-Risk of Linear Portfolios**

The VaR of linear portfolios can be evaluated analytically:
Bibliography


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