One Tailed Tests of Means for Multivariate Normal Distributions Derived by Generalized Geometric Programming

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Abstract

This paper studies multivariate one-tailed tests of the means, which occur in many application areas of statistics. The statistical distribution of the single-sided statistic is found using the union-intersection principle of S.N. Roy to formulate the estimation problem and generalized geometric programming to analyze and solve it. Generalized geometric programming is key to the solution as it converts the primal problem into a dual problem, which is effectively zero degree of difficulty and thus relatively easy to solve. The generalized t-statistic (GT) is developed. \( (GT)^2 \) is a generalization of the Hotelling \( T^2 \) statistic. This is based on a generalized F statistic, which can be found by solving an equation. Statistical tables are provided. The statistic is used to perform an hypothesis test on senility using the Wechsler Adult Intelligence Scale

Keywords. Hypothesis testing, fractional programming, generalized geometric programming

AMS subject classification: 90C25, 90C32, 97K80
2. Generalized Geometric Programming

Generalized geometric programming generalizes the duality theory of geometric programming to convex programming via the use of conjugate functions. It rests on the three pillars of linearity, separability and convexity, which are explicitly recognized in the formulation.

The theory of generalized geometric programming pairs the following two programs.

(CP) Minimize \( g_0(x_0) \)

subject to explicit constraints
\[
g_i(x_i) \leq 0, \quad i \in I
\]

implicit constraints
\[
x_0 \in C_0, \quad x_i \in C_i, \quad i \in I
\]

and cone condition
\[
x \in \chi \subseteq \mathbb{R}^n
\]

(CD) (the generalized geometric programming dual of CP):

Minimize \( g_0^*(x_0^*) + \sum_i g_i^*(x_i^*, \lambda_i^*) \)
subject to implicit constraints

\[ x_0^* \in C_0^*, (x_i^*, \lambda_i^*) \in C_i^*, \ i \in I, \]

and cone condition

\[ x^* \in \chi^* \subset \mathbb{R}^n \]

The relations between CP and CD are as follows: \( \chi \) is a closed convex cone. \( \chi^* \) is the dual cone of \( \chi \), i.e., \( \chi^* = \{ x^* | \langle x^*, x \rangle \geq 0, \forall x \in \chi \} \). \( I \) is the index of explicit constraints in program CP. \( x = (x_0, x_I) \) is the Cartesian product of vector \( x_0 \) and \( x_I \). \( x^* = (x_0^*, x_I^*) \) is similarly defined. \( x_I \) is the Cartesian product of vectors \( x_i \) of dimension \( n_i, i \in I \). \( x_i^* \) is similarly defined. \( \langle \cdot, \cdot \rangle \) denotes a finite dimensional inner product.

\[ [g_i(x_i), C_i], \ i \in \{0\} \cup I \] is a pair of closed convex function \( g_i \) with domain being the convex set \( C_i \subset \mathbb{R}^n \). \[ [g_i^*(x_i^*), C_i^*], \ i \in \{0\} \cup I \] is a pair of closed convex function \( g_i^* \) defined over the convex set \( C_i^* \) and is the conjugate transform of \( [g_i(x_i), C_i] \), i.e.,

\[ g_i^*(x_i^*) = \sup_{x_i \in C_i} \left( \langle x_i^*, x_i \rangle - g_i(x_i) \right) \]

and

\[ C_i^* = \{ x_i^* | \sup_{x_i \in C_i} \left( \langle x_i^*, x_i \rangle - g_i(x_i) \right) < \infty \} \]
\[ [g_i^+(x_i^*, \lambda_i^*), C_i^+] \], \quad i \in I \] is the positive homogeneous extension of \[ [g_i^*(x_i^*), C_i^*] \] with

\[
g_i^+(x_i^*, \lambda_i^*) = \begin{cases} \sup\langle x_i^*, x_i \rangle & \text{if } \lambda_i^* = 0 \text{ and } \sup\langle x_i^*, x_i \rangle < \infty \\
\lambda_i^* g_i(x_i^*/\lambda_i^*) & \text{if } \lambda_i^* > 0 \text{ and } x_i^*/\lambda_i^* \in C_i 
\end{cases}
\]

\[
C_i^+ = \left\{ (x_i^*, \lambda_i^*) \mid \lambda_i^* = 0 \text{ and } \sup_{x_i \in C_i} \langle x_i^*, x_i \rangle < \infty \right\} 
\cup \left\{ (x_i^*, \lambda_i^*) \mid \lambda_i^* > 0 \text{ and } x_i^*/\lambda_i^* \in C_i \right\}.
\]

It is interesting to note that the dual functions of all the explicit constraints in the primal (program CP) occur in the objective function in the dual (program CD). Under mild conditions concerning feasibility and relative interiors (Peterson [4]), the primal and the dual programs are related at optimality in the following manner:

\[
g_0(x_0) + g_i^0(x_i^*) + \sum_{i} g_i^+(x_i^*, \lambda_i^*) = 0
\]

\[
x_i^* \in \partial g_i(x_i^*), \\
\lambda_i^* > 0, \quad i \in I
\]

These optimality conditions allow an optimal point for one program to be calculated from an optimal point of its dual. Here \( \partial g_i(x_i) \) denotes the subgradient set of \( g_i \) at the point \( x_i \), i.e.,

\[
\partial g_i(x_i) = \left\{ x_i^* \mid g_i(x_i) + \langle x_i^*, z_i - x_i \rangle \leq g_i(z_i), \forall z_i \in C_i \right\}.
\]
### Table 1: Wechsler Adult Intelligence Scale Sample Means

<table>
<thead>
<tr>
<th>Subtest</th>
<th>Sample 1 No Senile Factor</th>
<th>Sample 2 Senile Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_1 = 37$</td>
<td>$N_2 = 12$</td>
</tr>
<tr>
<td>Information</td>
<td>12.57</td>
<td>8.75</td>
</tr>
<tr>
<td>Similarities</td>
<td>9.57</td>
<td>5.33</td>
</tr>
<tr>
<td>Arithmetic</td>
<td>11.49</td>
<td>8.50</td>
</tr>
<tr>
<td>Picture Completion</td>
<td>7.97</td>
<td>4.75</td>
</tr>
</tbody>
</table>

### Table 2: Pooled Sample Variance-covariance Matrix $S$

\[
\begin{bmatrix}
11.2553 & 9.4042 & 7.1489 & 3.3830 \\
9.4042 & 13.5318 & 7.3830 & 2.5532 \\
7.1489 & 7.3830 & 11.5744 & 2.6170 \\
3.3830 & 2.5532 & 2.6170 & 5.8085 \\
\end{bmatrix}
\]