Parametric Robustness

Abstract: Gain and time-constant factors shift the inverse process relative to its feedback controller. A new robustness plot cross-graphs these shift factors that take the loop to the stability boundary as a function of frequency.

Keywords: robustness plot, parameter shift factors, stability region, inverse process, deadtime.

1. INTRODUCTION

Process dynamics may be imperfectly known and may change over time. Production rate, feed composition, energy supply, wear, fouling, and the environment may cause the change. Feedback-loop robustness is an ability to maintain stability in the presence of process change or uncertainty. Robustness limits may be quantified as the smallest combination of probable parameter changes of specific process elements that can bring the nominally stable loop to its stability boundary. Often such a completely parameterized structural dynamic model of the process is either not available or difficult to analyze. Instead generic measures of unstructured uncertainty are commonly applied to nominally stable linearized input-output models of the process and controller.

Phase information in the frequency neighborhood of open-loop unity-gain crossings is critical for predicting feedback-loop stability. Effective delay contributes to phase but not amplitude at these frequencies. The effective delay time includes deadtime plus other high-frequent dynamics. Static nonlinearity also contributes to uncertainty. Zero-frequency gain, as distinct from amplitude in the critical frequency regions, is most easily measured or calculated but may have little impact on stability.

2. TRADITIONAL ROBUSTNESS

The Nyquist plot is a graph of the product of the open-loop process \( G(\jmath \omega) \) and feedback-controller \( C(\jmath \omega) \). Its polar amplitude-phase trajectory, as a function of frequency, demonstrates stability by passing to the right of the critical \((1, 180^\circ)\) point, provided no unstable open-loop poles are cancelled by zeros. Traditional robustness measures (Åström & Hägglund, 2005) include gain and phase margins, changes in gain and phase that would bring the loop trajectory to the stability-limit point.

This multiplicative model emphasizes the controlled measurement’s response \( y \) to process output noise \( n \) (multiplied by the sensitivity) or setpoint (and measurement noise) \( r \) (multiplied by the complimentary-sensitivity):

\[
y = \frac{1}{1+GC} n + \frac{1}{1+(GC)^{-1}} r.
\]  

(1)

Multiplicative compensation (for example stable-pole cancellation with a PID or model-feedback controller) may achieve fast setpoint tracking, provided the controller output does not limit, but may also provide poor load rejection.

Phase depends on frequency and time-constants. Small deadtime changes can cause large phase changes at high frequencies. This is particularly relevant for systems with multiple unity-gain crossings. Multiple crossings may occur if the process has a resonance, a recycle stream, or a lead (a zero), also if the controller has derivative action or has deadtime feedback (e.g. Smith Predictor, Internal Model, or Foxboro’s PIDr (Shinskey 1994, Hansen 2003) controllers).

At the stability limit:

\[
1 + (G + \delta G)C = 0,
\]  

(2a)

\[
\left( \frac{\delta G}{1+GC} \right) \left( \frac{\delta C}{C} \right) = -1.
\]  

(2b)

Maximizing magnitudes separately over frequency

\[
\left| \frac{1}{1+(GC)^{-1}(\jmath \omega)} \right|_{\text{max}} \cdot \left| \frac{\delta C(\omega)}{C(\omega)} \right|_{\text{max}} < 1.0
\]  

(2c)

assures stability. The maximum complementary-sensitivity magnitude is at least 1. This unrestricted-frequency test fails to demonstrate closed-loop robustness whenever the process contains deadtime uncertainty, because the max absolute process-change ratio is at least 2. A specified max abs. process-change ratio less than 1 (restricted to all frequencies where absolute complementary-sensitivity may be 1 or more) leads to an allowable maximum absolute complementary-sensitivity (circle) for use as a Nyquist-plot robustness constraint in a controller-design optimization.
3. PARAMETRIC ROBUSTNESS

Parameterized robustness is based on the sum of the inverse-process \( G(s)^{-1} \) and controller \( C(s) \), thus avoiding pole cancellation issues. This additive model emphasizes the controlled measurement’s response \( y \) to an unmeasured-load disturbance \( v \) at the process input.

\[
y = \frac{1}{G^{-1} + C} v. \tag{3}
\]

Additive compensation can achieve good load rejection and setpoint response as shown in Figure 4.

A gain factor \( k \) multiplies the process amplitude; a value greater than one shifts the inverse-process amplitude \( |kG(f\omega)|^{-1} \) downward relative to the controller \( |C(i\omega)| \) on a Bode plot. A time-constant factor \( f \) simultaneously multiplies all process time constants (lag, lead, integral, resonant period, and delay times). Significant time constants are often inversely proportional to production rate. A value of \( f \) greater than one shifts the inverse process amplitude and phase to the left, relative to the controller. The robustness plot uses log scales to graph \( k_{SB} \) vs. \( f_{SB} \); the values of \( k \) and \( f \) that bring the nominally stable loop to the stability boundary, as a function of (radian) frequency \( \omega \). A trajectory loop further away from the nominal \((1,1)\) point indicates another root has reached the stability limit.

4. DEADTIME EXAMPLE

Impending instability may not be apparent with a nominal Nyquist trajectory as is demonstrated by an example. Figure 1 is a polar Nyquist plot for the unity-gain pure-delay process \( G(s) = e^{-s\tau} \) and its “ideal” controller \( C(s) = (1 - e^{-s\tau})^{-1} \) (Hansen 2000). This model-feedback controller is an extreme example of a PIDr, a Smith Predictor, or an Internal Model controller. The polar Nyquist trajectory makes an infinite number of clockwise encirclements of the origin and none of the critical (1.180) point, keeping this critical point to its left as frequency increases. Thus the loop is stable. The gain margin is 2 (or 6 dB) and the phase margin is \( \pm 60^\circ \). The complementary-sensitivity magnitude is 1. Even though the Nyquist plot does not penetrate a max complementary-sensitivity circle, the loop is not robust.

Lack of robustness is shown by shifting the process in the special characteristic equation expressed as:

\[
C^{-1}(i\omega) + k_{SB}G(i f_{SB} \omega) = 0. \tag{4}
\]

The phase mismatch is \( \theta_{SB} \equiv (f_{SB} - 1) \tau \omega \). Using algebra and trigonometric identities it can be shown that:

\[
\theta_{SB} = \tan^{-1}\left(\frac{\sin \theta_{SB}}{1 - \cos \theta_{SB}}\right) = \tan^{-1}\left(\frac{1}{\tan \frac{\theta_{SB}}{2}}\right), \tag{5a}
\]

\[
k_{SB} = 2 \cos \theta_{SB}, \tag{5b}
\]

\[
f_{SB} = 1 + \frac{\theta_{SB}}{\tau \omega}. \tag{5c}
\]

When \( \theta_{SB} = 0^\circ \), \( k_{SB} = 2 \), the gain margin. When \( k_{SB} = 1.0, \theta_{SB} = \pm 60^\circ \), the phase margin. However, \( f_{SB} \) approaches 1 as frequency increases, indicating a diminishing stability region.

Figure 1c is the robustness plot, \( k_{SB} \) vs. \( f_{SB} \), using logarithmic scales. The robustness trajectory proceeds from the lower right as frequency increases, encircling counterclockwise the nominal \((1,1)\) point with ever narrowing loops.

5. PID CONTROL

It is convenient to shift a PID (proportional \( P \), Integral \( I \), derivative \( D \)) controller:

\[
C(s) = \frac{1}{p} \left(\frac{1}{s} + 1 + Ds\right), \tag{7}
\]

relative to the inverse process in order to locate the robustness boundary:

\[
k_{SB} \frac{D}{f_{SB}} \left(\frac{1}{\omega} + 1 + \frac{D\omega}{f_{SB}}\right) = \left|G^{-1}(i\omega)\right|, \tag{8}
\]

where \( \omega = f_{SB}\omega \).

Equating reals and imaginaries to 0:

\[
k_{SB} = -\frac{1}{P} \text{Re}\left[G^{-1}(i\omega)\right]. \tag{9a}
\]